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Instability of Renormalization-Group Pathologies Under Decimation

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We investigate the stability and instability of pathologies of renormalization group transformations for lattice spin systems under decimation. In particular we show that, even if the original renormalization group transformation gives rise to a non-Gibbsian measure, Gibbsianness may be restored by applying an extra decimation transformation. This fact is illustrated in detail for the block spin transformation applied to the Ising model. We also discuss the case of another non-Gibbsian measure with nicely decaying correlations functions which remains non-Gibbsian after arbitrary decimation.

KEY WORDS: Renormalization group; decimation; non-Gibbsianness; Ising model.

1. INTRODUCTION

In this note we discuss some aspects of the problem of defining, on rigorous grounds, a renormalization group transformation (RGT) for the Gibbs measure of lattice spin systems of statistical mechanics. For simplicity and without a true loss of generality (see the end of Section 2), we confine our attention to the average block spin transformation for the 2D Ising model at low temperature and large positive external field.

In the basic reference[(4)] the authors discuss in a very complete and clear way the possible pathologies that may arise when applying a RGT to a perfectly well-behaved Gibbs measure like the one above. To be more

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specific, suppose that μ denotes the starting Gibbs measure on a probability space Ω and that

$$\nu = T_b \mu \tag{1.1}$$

denotes the transformed measure, obtained by applying to μ a RGT T_b acting on "scale" b and defined on a new probability space Ω'. The system described by the measure μ and with configurational variables with values in Ω is called the "object" system, whereas the system described by ν with configurational variables with values in Ω' is called the "image" system.

As the authors of ref. 4 point out, the main (rather surprising) pathology of the above RGT is that the renormalized measure ν can very well be *non-Gibbsian*, that is, the associated system of conditional probabilities is not compatible with any finite-norm potential. That may happen even if the starting measure μ, e.g., the unique Gibbs measure of some finite-range interaction, has all the nice properties describing the one-phase region: analyticity, exponential decay of correlations, convergent cluster expansion, etc.

In ref. 4 one can find many examples of such pathology. Moreover, the same authors show that the typical mechanism behind the *non-Gibbsianness* of the measure ν is the appearance of long-range order, that is, a phase transition, in the object system *conditioned* to some particular configurations of the image system. Such long-range order implies, in particular, that the measure ν violates a necessary condition for being Gibbsian, namely "quasilocality" of its family of conditional probabilities $\{\pi_\Lambda\}_\Lambda$. Such a condition, introduced by Kozlov (see ref. 8 and Theorem 2.12 in ref. 4), roughly speaking implies some sort of uniform continuity of the conditional probabilities π_Λ with respect to the conditioning configuration (see also refs. 1 and 5 for a critical discussion).

An interesting example given in ref. 4 of the above phenomenon refers to the "decimation transformation" on scale b, T_b^d, applied to the Gibbs measure $\mu_{\beta,h}$ of the Ising model at low temperature, $\beta \gg 1$, and small magnetic field h. Such a transformation associates to the original measure $\mu_{\beta,h}$ its marginal (or relativization) on the spin variables sitting on the sites of the sublattice $\mathbf{Z}^d(b)$ of $\mathbf{Z}^d$ with spacing b. In other words, one integrates out all the variables in $\mathbf{Z}^d \backslash \mathbf{Z}^d(b)$.

In ref. 4 it was proved that, for any given b and for suitable values of β and h, the measure

$$\nu = T_b^d \mu_{\beta,h} \tag{1.2}$$

is *non-Gibbsian*.

This fact is the consequence of the degeneracy of the ground state of the Ising model restricted to $\mathbf{Z}^d \backslash \mathbf{Z}^d(b)$ if the conditioning spins at the sites of $\mathbf{Z}^d(b)$ are held fixed in some suitable, particular configuration, for instance, all the spins equal to -1, and the value of the magnetic field h is suitably chosen as a function of the lattice spacing b. Such a degeneracy, using the theory of Pirogov and Sinai, leads to a first-order phase transition at low enough temperature for the same constrained system.

One may say that the above "spurious" phase transition comes from the fact that, on a too short length scale b (with respect to the thermodynamic parameters and mainly to h), the system is reminiscent of the phase transition taking place at $h=0$. It thus appears plausible that the above pathology could be eliminated and therefore Gibbsianness recovered by choosing a large enough spacing b for given fixed values of β and h; in particular, it should be sufficient to iterate a sufficiently large number of times the same transformation in order to come back to the space of Gibbsian measures. This is what we actualy proved in ref. 10 together with some additional results such as convergence of a cluster expansion for ν and the convergence of $(T_b^d)^n \mu_{\beta,h}$ to a trivial fixed point as $n \to \infty$.

In ref. 4 there is another, more subtle, example of pathology, referring to the so-called "block averaging transformation" for the 2D Ising model. This example and the associated pathology are the main object of the present note.

The transformation is defined as follows. Suppose we partition the lattice $\mathbf{Z}^2$ into 2×2 blocks Q_i and let us denote by m_i one of the five possible values of the magnetization (sum of the spins) inside the block Q_i. Then the transformed measure

$$\mu^B(\{m_i\}) = T_2^B \mu_{\beta,h}$$

is defined simply as the probability distribution of the variables $m \equiv \{m_i\}$.

Here the violation of quasilocality and thus the non-Gibbsianness of $\mu^B(\{m_i\})$ are due to the presence, for large enough β and arbitrary value of h, of a first-order phase transition in the multicanonical model represented by the object system constrained to have zero magnetization in each block Q_i. Notice that, since the local magnetizations are fixed, the value of the magnetic field is irrelevant. Although the proof of this result was given only for the case of 2×2 blocks, it persists for any even value of the side of the blocks Q_i, thus excluding the possibility of restoring Gibbsianness by simply enlarging the side of the blocks, in contrast to what happens for the decimation transformation.

On the other hand, if, for example, the magnetic field is large, the object system without constraints is very close to a product measure and the non-Gibbsian measure $\mu^B(\{m_i\})$ itself enjoys nice mixing properties such as exponential decay of truncated correlation functions and, quite likely, a weaker version of quasilocality, such as the one introduced in ref. 5. Moreover, it is quite obvious that the event of having zero magnetization in each block is exceptional and thus, in some sense, the above pathology should be "unstable" with respect to a little bit of decimation at least if the surviving variables are separated by a distance larger than the correlation length of the original system.

This kind of consideration was already suggested, on a informal level, in ref. 4 (see p. 1066).

In the present note we pursue the above point of view quite seriously, since, in our opinion, the "stability" or "instability" of non-Gibbsianness of a measure under decimation is a relevant property. Decimation in fact corresponds to select certain variables, which are the only "relevant" ones for the kind of question in which one is interested, and disregard (integrate out) the "irrelevant" ones; moreover, important thermodynamic quantities such as the free energy (and their analyticity properties) or the asymptotic behavior of the truncated correlations can be computed equally well with the decimated measure. Thus, if Gibbsianness can be restored with the help of some decimation, then the pathologies described above become irrelevant at least as far as certain variables are concerned. On the contrary, if the measure μ under consideration is non-Gibbsian and remains of such a type after an arbitrary decimation, then such a character becomes, in our opinion, a much more important feature of the system described by μ, probably related to some nontrivial long-range dependence hidden inside the system itself.

In this note we illustrate in full detail the above considerations, first for the block spin Ising model (see Section 2) and then for the invariant measure of a certain stochastic dynamics on the configuration space $\{0, 1\}^{\mathbb{Z}^2}$ (see Section 3). In the first case we prove that, if we decimate the block spin model on the even blocks (see Section 2 for details) and we take the external field h large enough, then we end up with a nice, weakly coupled Gibbs measure whose potential is expressed via a convergent cluster expansion. To perform the calculation we use the following property, which in the sequel will be referred to as commutativity of the block decimation with the block spin transformation. The two sequences of transformations acting on the original Gibbs measure give rise to the same measure:

1. First decimate over the even 2×2 blocks; then perform the block averaging over the surviving blocks.

2. First perform the block averaging over all the 2×2 blocks; then apply the standard decimation over the block-spin variables associated to the even blocks.

In the second case we show that, independently of the side of the blocks of the decimation, the decimated measure remains non-Gibbsian like the starting measure.

After the present paper was completed, we learnt of a recent work by A. v. Enter, R. Fernández, and R. Kotecký where, in particular, the authors establish non-Gibbsianness of the renormalized measure $\nu = T_b^{mr} \mu_{\beta,h}$ obtained by applying the majority rule transformation, over blocks of side b, to the Ising Gibbs measure $\mu_{\beta,h}$ for β and h large enough.

One can easily check that, by the same methods developed in Section 2 of the present paper, it is possible to restore Gibbsianness by simply decimating the measure ν over the even blocks (see Section 2). In other words, a statement analogous to the one of Theorem 2.1 holds true.

2. THE BLOCK SPIN AND DECIMATION TRANSFORMATIONS

In this main section we discuss in detail the effect of a decimation over the odd blocks (see below) on the block spin Ising model for which *non-Gibbsianness* was proved in ref. 4. For simplicity we restrict ourselves to the case of large external magnetic field (but see the remark at the end of the section for more general situations).

We show that, after the decimation, Gibbsianness is recovered. As already explained in the introduction, a key remark is that the two transformations, the block spin and the decimation, "commute," as discussed before, so that we can first decimate the original Gibbs measure of the Ising model and then do the block average transformation. The technical tool is the cluster expansion that provides naturally all the necessary cancellations. Let us start with the details.

The Ising Hamiltonian in a volume $\Lambda \in \mathbf{Z}^d$ with open (empty) or periodic boundary conditions is given by

$$H_\Lambda(\sigma_\Lambda) = -J/2 \sum_{x,\, y \in \Lambda} \sigma_x \sigma_y - h/2 \sum_{x \in \Lambda} \sigma_x \tag{2.1}$$

where $\sigma_\Lambda \in \Omega_\Lambda \equiv \{-1, +1\}^\Lambda$ and h is the external magnetic field.

The corresponding finite-volume Gibbs measure at inverse temperature β and magnetic field h is given by

$$\mu_\Lambda = \frac{\exp[-\beta(H_\Lambda(\sigma_\Lambda))]}{Z_\Lambda} \tag{2.2}$$

where the normalization factor

$$Z_\Lambda = \sum_{\sigma_\Lambda \in \Omega_\Lambda} \exp[-\beta(H_\Lambda(\sigma_\Lambda))] \tag{2.3}$$

is called the *partition function.*

We will consider the case when both the inverse temperature β and the external magnetic field h are very large.

For the sake of simplicity of the exposition we will assume that the dimension is $d=2$.

Consider the partition of $\mathbf{Z}^2$ into 2×2 square blocks Q_i of side 2 (each containing four sites). A block Q_i can be characterized by its leftmost down site $x(Q_i)$. The $x(Q_i)$ are of the form

$$\begin{gathered} x(Q_i)\equiv x^{(i)}\equiv (x_1^{(i)}, x_2^{(i)}) \\ x_1^{(i)}=2y_1^{(i)}, \qquad x_2^{(i)}=2y_2^{(i)} \qquad \text{with} \quad y^{(i)}\equiv (y_1^{(i)}, y_2^{(i)})\in \mathbf{Z}^2 \end{gathered} \tag{2.4}$$

We write

$$Q_i=Q(y^{(i)}) \qquad \text{if} \quad y^{(i)}=x(Q_i)/2 \tag{2.5}$$

Now we introduce a partition of the lattice $\mathbf{Z}^2$ into two sublattices $\mathbf{Z}^2_{\mathrm{e}}$ and $\mathbf{Z}^2_{\mathrm{o}}$ (the subscripts e and o, respectively, stand for even and odd). They are given by

$$\mathbf{Z}^2_{\mathrm{e,o}}=\{y\equiv (y_1, y_2)\in \mathbf{Z}^2\colon y_1+y_2=\text{even integer, odd integer}\} \tag{2.6}$$

Given a 2×2 block Q_i we call it even or odd according to the sublattice to which $y^{(i)}=x(Q_i)/2$ belongs.

We decompose the original lattice $\mathbf{Z}^2$ into the union

$$\mathbf{Z}^2=\mathscr{A}\cup\mathscr{B} \tag{2.7}$$

where $\mathscr{A}=\bigcup_i A_i$ is the set of the even blocks A_i with $x(A_i)/2\in \mathbf{Z}^2_{\mathrm{e}}$ and $\mathscr{B}=\bigcup_i B_i$ is the set of the odd blocks B_i with $x(B_i)/2\in\mathbf{Z}^2_{\mathrm{o}}$. Notice that in our notation we suppose that the total set $\mathscr{Q}=\bigcup_i Q_i$ of the 2×2 blocks as well as the sets $\mathscr{A}=\bigcup_i A_i$ and $\mathscr{B}=\bigcup_i B_i$ of even and odd blocks, respectively, is given a certain order, for example, the lexicographic one, but this ordering will never be used explicitly.

We use the notation α_i, β_i to denote, respectively, the spin configurations inside the blocks A_i, B_i; α_i, β_i take $2^4=16$ possible values.

We denote by e_1, e_2, e_3, e_4 the unit vectors $(0,1)$, $(1,0)$, $(-1,0)$, $(0,-1)$.

Given a block $B_i\equiv Q_{l(i)}$ with $x(Q_{l(i)})/2=y^{(l(i))}$, we denote by A_i^1, A_i^2, A_i^3, A_i^4 the four nearest-neighbor A-blocks given by $A_i^j=Q(y^{(l(i))}+e_j)$, $j=1,\dots,4$, and by α_i^1, α_i^2, α_i^3, α_i^4 the corresponding spin configurations. We use $\underline{\alpha}_i$ to denote the set of spin configurations α_i^1, α_i^2, α_i^3, α_i^4 in these four A-blocks.

By $Z_{B_i}^{\underline{\alpha}_i}$ we denote the partition function in the block B_i with boundary conditions given by $\underline{\alpha}_i$.

We call 0 a fixed reference configuration inside a 2×2 block; for instance, 0 can be chosen to be the configuration with all minus spins inside the block. We use $\underline{0}$ to denote the configuration $\underline{\alpha}_i \equiv \alpha_i^j = 0$ $\forall j = 1,..., 4$ in the set A_i^1, A_i^2, A_i^3, A_i^4 of nearest-neighbor A-blocks to a given B-block B_i.

Given an integer L multiple of 4, consider the squared box $\Lambda \equiv \Lambda_L \equiv [-L/2, L/2+1]^2$.

We choose, for simplicity, periodic boundary conditions; namely Λ, by identifying its opposite sides, becomes a two-dimensional finite torus. Any other boundary condition could be considered as well, with only minor changes.

Using simply α, β to denote the global spin configuration in all the A-blocks, B-blocks, respectively, contained in Λ, we can write te following expression for the partition function in Λ (with periodic boundary conditions):

$$Z_\Lambda = \sum_\alpha Z_\Lambda(\alpha) \tag{2.8}$$

where

$$Z_\Lambda(\alpha) = \exp\left(\sum_{A_i \subset \Lambda} H(\alpha_i)\right) \prod_{B_i \subset \Lambda} Z_{B_i}^{\underline{\alpha}_i} \tag{2.9}$$

and, for $\alpha_i = \sigma_{A_i}$, $H(\alpha_i)$ is the self-energy inside the block A_i:

$$H(\alpha_i) \equiv H_{A_i}(\sigma_{A_i}) = -J/2 \sum_{x,\, y \in A_i} \sigma_x \sigma_y - h/2 \sum_{x \in A_i} \sigma_x \tag{2.10}$$

Expression (2.8) is simply obtained by a decimation procedure; namely by first summing over the β-variables keeping fixed the α-variables, which play the role of fixed boundary conditions. The sum over the β-variables, for fixed α, is immediately seen to factorize into independent sums over the single β_i which are mutually decoupled because of the form (nearest neighbor) of the Ising interaction.

By simple manipulations of the previous expression we get

$$\begin{aligned} Z_\Lambda = &\prod_{B_i \subset \Lambda} \left(\frac{1}{Z_{B_i}^{\underline{0}}}\right)^3 \sum_\alpha \exp\left(\sum_{A_i \subset \Lambda} H(\alpha_i)\right) \\ &\times \prod_{B_i \subset \Lambda} \left(\frac{Z_{B_i}^{\alpha_i^1,\alpha_i^2,\alpha_i^3,\alpha_i^4} Z_{B_i}^{\underline{0}} Z_{B_i}^{\underline{0}} Z_{B_i}^{\underline{0}}}{Z_{B_i}^{\alpha_i^1,0,0,0} Z_{B_i}^{0,\alpha_i^2,0,0} Z_{B_i}^{0,0,\alpha_i^3,0} Z_{B_i}^{0,0,0,\alpha_i^4}} - 1 + 1\right) \\ &\times Z_{B_i}^{\alpha_i^1,0,0,0} Z_{B_i}^{0,\alpha_i^2,0,0} Z_{B_i}^{0,0,\alpha_i^3,0} Z_{B_i}^{0,0,0,\alpha_i^4} \end{aligned} \tag{2.11}$$

and we define

$$\tilde{Z}_A = \prod_{B_i \subset A} \left(\frac{1}{Z^0_{B_i}}\right)^3$$
$$\times \prod_{A_i \subset A} \sum_{\alpha_i} (\exp(H(\alpha_i)\, Z^{\alpha_i,0,0,0}_{B_i^3} Z^{0,\alpha_i,0,0}_{B_i^4} Z^{0,0,0,\alpha_i}_{B_i^2} Z^{0,0,\alpha_i,0}_{B_i^1}) \tag{2.12}$$

where if $A_i \equiv Q_{m(i)}$, we set $B_i^j = Q(y^{m((i))} + e_j)$, $j = 1,..., 4$, and we use the short forms $\sum_{\alpha_i}$, $\sum_\alpha$ to denote $\sum_{\alpha_i \in \Omega_{A_i}}$, $\sum_{\alpha \in \Omega_{\mathscr{A}}}$, respectively.

A useful graphical way to describe the r.h.s. of (2.12) is to associate to any one of the four partition functions appearing in the r.h.s. of (2.11), namely to $Z^{\alpha_i^1,0,0,0}_{B_i}$, $Z^{0,\alpha_i^2,0,0}_{B_i}$, $Z^{0,0,\alpha_i^3,0}_{B_i}$, $Z^{0,0,0,\alpha_i^4}_{B_i}$, four arrows, emerging from the block B_i and ending in the blocks A_i^1, A_i^2, A_i^3, A_i^4; namely four arrows parallel to the four unit vectors e_1, e_2, e_3, e_4, respectively. Then in (2.12) there appear the terms (partition functions) corresponding to the four arrows ending into the A-block A_i and emerging from the four nearest-neighbor B-blocks.

Consider now, for every A-block A_i, the probability measure on α_i given by

$$\nu(\alpha_i) = \frac{\exp(H(\alpha_i))\, Z^{\alpha_i,0,0,0}_{B_i^3} Z^{0,\alpha_i,0,0}_{B_i^4} Z^{0,0,\alpha_i,0}_{B_i^2} Z^{0,0,0,\alpha_i}_{B_i^1})}{\sum_{\alpha_i} \exp(H(\alpha_i))\, Z^{\alpha_i,0,0,0}_{B_i^3} Z^{0,\alpha_i,0,0}_{B_i^4} Z^{0,0,\alpha_i,0}_{B_i^2} Z^{0,0,0,\alpha_i}_{B_i^1})} \tag{2.13}$$

We set

$$\hat{Z}_{A_i} = \sum_{\alpha_i} \exp(H(\alpha_i))\, Z^{\alpha_i,0,0,0}_{B_i^3} Z^{0,\alpha_i,0,0}_{B_i^4} Z^{0,0,\alpha_i,0}_{B_i^2} Z^{0,0,0,\alpha_i}_{B_i^1}) \equiv Z^0_{V_i} \tag{2.14}$$

where

$$V_i = A_i \cup B_i^1 \cup B_i^2 \cup B_i^3 \cup B_i^4 \tag{2.15}$$

From (2.12), (2.14) we obtain

$$\tilde{Z} = \prod_{B_i \subset A} \left(\frac{1}{Z^0_{B_i}}\right)^3 \prod_{A_i \subset A} \hat{Z}_{A_i} \tag{2.16}$$

from (2.11), (2.13), (2.14), and (2.16), we get

$$Z_A = \tilde{Z}_A \sum_\alpha \prod_{A_j \subset A} \nu(\alpha_j) \prod_{B_i \subset A} (\psi_{B_i}(\alpha_i^1, \alpha_i^2, \alpha_i^3, \alpha_i^4) + 1) \tag{2.17}$$

where

$$\psi_{B_i}(\alpha_i^1, \alpha_i^2, \alpha_i^3, \alpha_i^4) \equiv \frac{Z_{B_i}^{\alpha_i^1,\alpha_i^2,\alpha_i^3,\alpha_i^4} Z_{B_i}^0 Z_{B_i}^0 Z_{B_i}^0}{Z_{B_i}^{\alpha_i^1,0,0,0} Z_{B_i}^{0,\alpha_i^2,0,0} Z_{B_i}^{0,0,\alpha_i^3,0} Z_{B_i}^{0,0,0,\alpha_i^4}} - 1 \tag{2.18}$$

We define the *renormalized Hamiltonian* as

$$H_\Lambda^r = \sum_{A_i \subset \Lambda} H(\alpha_i) + \sum_{B_i \subset \Lambda} -\log Z_{B_i}^{\alpha_i^1,\alpha_i^2,\alpha_i^3,\alpha_i^4} \tag{2.19}$$

Then after having extracted the one-body part we get

$$\begin{aligned} H_\Lambda^r = {} & \sum_{A_i \subset \Lambda} H(\alpha_i) + \sum_{B_i \subset \Lambda} -(\log Z_{B_i}^{\alpha_i^1,0,0,0} + \log Z_{B_i}^{0,\alpha_i^2,0,0} \\ & + \log Z_{B_i}^{0,0,\alpha_i^3,0} + \log Z_{B_i}^{0,0,0,\alpha_i^4}) \\ & + \sum_{B_i \subset \Lambda} -\log\left(\frac{Z_{B_i}^{\alpha_i^1,\alpha_i^2,\alpha_i^3,\alpha_i^4} Z_{B_i}^0 Z_{B_i}^0 Z_{B_i}^0}{Z_{B_i}^{\alpha_i^1,0,0,0} Z_{B_i}^{0,\alpha_i^2,0,0} Z_{B_i}^{0,0,\alpha_i^3,0} Z_{B_i}^{0,0,0,\alpha_i^4}}\right) + \text{const} \end{aligned} \tag{2.20}$$

Now we want to use the expression given in (2.17) to make the second step; namely the sum over the spin configuration $\alpha_i \equiv \sigma_{A_i}$ with given values m_i of the magnetization $m_{A_i} \equiv m_{A_i}(\alpha_i) \equiv \sum_{x \subset A_i} \sigma_x$ in the blocks A_i.

If $N = N(L)$ is the total number of A-blocks in Λ, let

$$Z_\Lambda(m_1,..., m_N) = \sum_\alpha \prod_{A_i \subset \Lambda} (\mathbf{1}_{m_{A_i} = m_i}(\alpha_i))\, Z_\Lambda(\alpha) \tag{2.21}$$

where

$$\begin{aligned} \mathbf{1}_{m_{A_i} = m_i}(\alpha_i) = 1 \qquad & \text{if} \quad m_{A_i} \equiv m_{A_i}(\alpha_i) = m_i \\ \mathbf{1}_{m_{A_i} = m_i}(\alpha_i) = 0 \qquad & \text{otherwise} \end{aligned} \tag{2.22}$$

and $Z_\Lambda(\alpha)$ has been defined in (2.9)).

Now we transform our original block spin system, described by (2.21), into a polymer system; for this purpose we need some definitions.

A (four-body) *bond* $p_i \equiv p(B_i)$, for a given B_i, is the set $p_i = \{A_i^1, A_i^2, A_i^3, A_i^4\}$ of the four A-blocks nearest neighbor to B_i.

Its *support* $\tilde{p}_i$ is given by $\tilde{p}_i = \bigcup_{j=1}^4 A_i^j$.

A set of bonds $R = p_i,..., p_k$ is called a *polymer* if it is *connected* in the sense that for every pair $p_i, p_j \in R$ there exists a chain $p_{k_1},..., p_{k_l} \in R$, with $p_{k_1} = p_i$, $p_{k_l} = p_j$ of bonds which are connected in the sense that $\tilde{p}_{k_i} \cap \tilde{p}_{k_{i+1}} \neq \emptyset$, $i = 1,..., l$.

We call a *support* of a polymer R and we denote by $\tilde{R}$ the union of the supports $\tilde{p}_i$ of the bonds $p_i \in R$.

We say that two polymers R_1 and R_2 are *compatible* if $\tilde{R}_1 \cap \tilde{R}_2 = \varnothing$.

We set

$$\tilde{\zeta}_R(\alpha_R) = \prod_{p(B_i) \in R} \psi_{B_i}(\alpha_i^1, \alpha_i^2, \alpha_i^3, \alpha_i^4) \tag{2.23}$$

Now let

$$\bar{Z}_\Lambda(m_1,..., m_N) = \prod_{\Lambda_i \subset \Lambda} \bar{Z}_{\Lambda_i}(m_i) \tag{2.24}$$

with

$$\bar{Z}_{\Lambda_i}(m_i) = \sum_{\alpha_i} \nu(\alpha_i)\, \mathbf{1}_{m_{\Lambda_i} = m_i}(\alpha_i) \tag{2.25}$$

Let

$$\mu_{m_i}(\alpha_i) \equiv 1/\bar{Z}_{\Lambda_i}(m_i)\, \nu(\alpha_i)\, \mathbf{1}_{m_{\Lambda_i} = m_i}(\alpha_i) \tag{2.26}$$

From (2.8), (2.9), (2.11), (2.17), (2.21), (2.24), (2.25) it is easy to get

$$\frac{Z_\Lambda(m_1,..., m_N)}{\tilde{Z}_\Lambda \bar{Z}_\Lambda(m_1,..., m_N)} = \prod_{\Lambda_i \subset \Lambda} \sum_{\alpha_i} \mu_{m_i}(\alpha_i) \left(1 + \sum_{n \geqslant 1} \sum_{\substack{R_1,..., R_n: \\ \tilde{R}_i \subseteq \Lambda, \tilde{R}_i \cap \tilde{R}_j = \varnothing}} \prod_{i=1}^n \tilde{\zeta}_{R_i}(\alpha_{R_i})\right) \tag{2.27}$$

Now, if we introduce the *activity* $\zeta(R) \equiv \zeta_m(R)$ of a generic polymer R as

$$\zeta(R) = \prod_{\Lambda_i \in \tilde{R}} \sum_{\alpha_i} \mu_{m_i}(\alpha_i)\, \tilde{\zeta}_R(\alpha_R) \tag{2.28}$$

we can write

$$\frac{Z_\Lambda(m_1,..., m_N)}{\tilde{Z}_\Lambda \bar{Z}_\Lambda(m_1,..., m_N)} = 1 + \sum_{n \geqslant 1} \sum_{\substack{R_1,..., R_n: \\ \tilde{R}_i \subseteq \Lambda, \tilde{R}_i \cap \tilde{R}_j = \varnothing}} \prod_{i=1}^n \zeta(R_i) \tag{2.29}$$

Now we observe that in the region of thermodynamic parameters that we are considering, namely h and β large, the activity of our polymers is indeed very small, uniformly in the m_i, in the proper sense so that we can apply the theory of the cluster expansion and obtain its convergence. As we will discuss later, one could simply assume $h \neq 0$ and β sufficiently large provided the 2×2 blocks are replaced by blocks of large enough side (depending on h and β).

In our case of h very large everything is much simpler since after decimation our system is weakly coupled on scale one.

In particular it is easy to get the following statement:

Proposition 2.1. For every $\varepsilon > 0$ there exists a value $h(\varepsilon)$ of the magnetic field such that if $h > h(\varepsilon)$ and β is sufficiently large, then

$$\sup_{\alpha_i^1, \alpha_i^2, \alpha_i^3, \alpha_i^4} |\psi_{B_i}(\alpha_i^1, \alpha_i^2, \alpha_i^3, \alpha_i^4)| < \varepsilon \tag{2.30}$$

From the previous proposition one can easily deduce, by standard methods, the convergence of a cluster expansion of the thermodynamic as well as the correlation functions. The elementary geometric objects of this expansion will be clusters of incompatible polymers (see, for instance, ref. 6 for more details). Many properties for the image block-spin system (after decimation over the odd B-blocks), described by the probability measure with weights proportional to $Z_\Lambda(m_1,..., m_N)$, can be deduced using this convergent cluster expansion.

We will concentrate now on one important feature of this measure, namely its *Gibbsianness*.

First of all let us introduce the doubly renormalized Hamiltonian, after decimation on the odd B-blocks and block-averaging over the surviving even A-blocks. We will denote it by $H_\Lambda^{b,d}(m_1,..., m_N)$, where in the superscript b, d stand for block spin averaging transformations and for decimation transformation, respectively.

We define it as

$$H_\Lambda^{b,d}(m_1,..., m_N) = -\log Z_\Lambda(m_1,..., m_N) \tag{2.31}$$

which corresponds to a particular choice of the zero of the energy for our doubly renormalized system (namely obtained by decimation and block spin transformation).

Our doubly renormalized probability measure is

$$\begin{aligned} \mu_\Lambda^{b,d}(m_1,..., m_N) &= \frac{Z_\Lambda(m_1,..., m_N)}{\sum_{m_1,..., m_N} Z_\Lambda(m_1,..., m_N)} \\ &\equiv \frac{\exp(-H_\Lambda^{b,d}(m_1,..., m_N))}{\sum_{m_1,..., m_N} \exp(-H_\Lambda^{b,d}(m_1,..., m_N))} \end{aligned} \tag{2.32}$$

We now state and prove our main result about the Gibbsianness of our doubly renormalized measure $\mu^{b,d}$ in the thermodynamic limit $\Lambda \to \mathbf{Z}^2$.

Theorem 2.1. If the external magnetic field h and the inverse temperature β are sufficiently large:

(i) There exists the (weak) thermodynamic limit of the finite-volume, doubly renormalized measure

$$\mu^{b,d}(m_1,..., m_N) = \lim_{\Lambda \to \mathbf{Z}^2} \mu_\Lambda^{b,d}(m_1,..., m_N) \tag{2.33}$$

(ii) $\mu^{b,d}(m_1,..., m_N)$ is a Gibbs measure corresponding to a finite-norm, translationally invariant, potential.

Proof. The proof of the theorem is based on the theory of cluster expansion applied to the system of polymers described by (2.29). For this purpose let us recall a proposition which summarizes the basic results on cluster expansions that we need in order to prove Gibbsianness.

The proof of the proposition together with more details can be found in ref. 2.

Proposition 2.2. Let Ξ_Λ denote the polymer partition function:

$$\Xi_\Lambda = 1 + \sum_{k \geqslant 1} \sum_{\substack{R_1,..., R_k: \\ \bar{R}_i \subseteq \Lambda, 1 \leqslant i \leqslant k, \\ \bar{R}_i \cap \bar{R}_{i'} = \varnothing, 1 \leqslant i < i' \leqslant k}} \prod_{j=1}^{k} \zeta(R_j) \tag{2.34}$$

Suppose that $\zeta(\cdot)$ is translationally invariant and that there exist two positive constants σ and g_p such that the following estimate holds:

$$|\zeta(R)| \leqslant \sigma^{|\bar{R}|} \prod_{p \in R} g_p \tag{2.35}$$

Let $\mathscr{P}$ be the set of all the bonds in the whole lattice and let κ be given by

$$\kappa = \sum_{p \in \mathscr{P}: p \ni A_0} g_p \tag{2.36}$$

Then, if the following bound holds:

$$\exp(\kappa) < \frac{1}{\sqrt{\sigma}(2 - \sqrt{\sigma})} \tag{2.37}$$

we have

$$\sum_{R: A_0 \subset \bar{R} \subseteq \Lambda} |\zeta(R)| \leqslant \sigma\kappa \frac{[1 - (e^\kappa - 1)]}{1 + \sigma^2 e^\kappa - 2\sigma e^\kappa} \equiv G(\kappa, \sigma) \tag{2.38}$$

uniformly in Λ, and if $\mathscr{R}$ is the set of all the polymers in the whole lattice, for any polymer $R \in \mathscr{R}$,

$$\sum_{k \geqslant 1} \sum_{\substack{R_1,..., R_k \in \mathscr{R}: \\ R_1 = R}} |\varrho(R_1,..., R_k)| \prod_{i=1}^{k} |\zeta(R_i)| \leqslant |\zeta(R)| \frac{\exp[G(\kappa, \sqrt{\sigma}) |R|]}{1 - \sqrt{\sigma} \exp[G(\kappa, \sqrt{\sigma})]} \tag{2.39}$$

Here ϱ denotes the standard Möbius function:

$$\varrho(R_1,..., R_k) = \frac{1}{k!} \sum_{g \in \mathbf{G}(R_1,..., R_k)} (-1)^{\#(\text{edges in } g)} \tag{2.40}$$

where $\mathbf{G}(R_1,..., R_k)$ stands for the set of all connected subgraphs of the graph with k vertices $\{1,..., k\}$ and with the edges corresponding to those pairs (i, j) for which $\tilde{R}_i \cap \tilde{R}_j \neq \varnothing$ [the sum in (2.40) equals zero if $\mathbf{G}(R_1,..., R_k)$ is empty and one if $k = 1$].

By using Proposition 2.2, we can control the convergence of the standard cluster representation

$$\Xi_\Lambda = \exp\left[\sum_{k \geqslant 1} \sum_{\substack{R_1,..., R_k: \\ \tilde{R}_i \subseteq \Lambda, 1 \leqslant i \leqslant k}} \varrho(R_1,..., R_k) \prod_{j=1}^{k} \zeta(R_j)\right] \tag{2.41}$$

which follows from (2.34), (2.38), (2.39).[6]

Let us now conclude the proof of the Theorem. Part (i) easily follows from Propositions 2.1 and 2.2 and standard methods of the theory of cluster expansion.[6]

To get part (ii) we start from the following expression for the Hamiltonian $H_\Lambda^{b,d}$:

$$H_\Lambda^{b,d} = \sum_{k \geqslant 1} \sum_{\substack{R_1,..., R_k: \\ \tilde{R}_i \subseteq \Lambda, 1 \leqslant i \leqslant k}} \varrho(R_1,..., R_k) \prod_{j=1}^{k} \zeta(R_j) - \sum_{A_i \subset \Lambda} \log \bar{Z}_{A_i}(m_i) \tag{2.42}$$

which follows from (2.29), (2.31), and (2.41). We recall that the activities $\zeta(R) \equiv \zeta_{m_R}(R)$ depend on the fixed values of $m_R \equiv \{m_i\}$ with i such that $A_i \subset \tilde{R}$.

The potential Ψ can be obtained in the following way. Given a set Γ of A blocks, $\Gamma = \{A_{i_1} \cdots A_{i_k}\}$ and $m_\Gamma = \{m_{i_1} \cdots m_{i_k}\}$, $K > 1$, we define

$$\Psi_\Gamma(m_\Gamma) = \sum_{j \leqslant k} \sum_{\substack{R_1,..., R_j: \\ \bigcup \tilde{R}_j = \Gamma}} \varrho(R_1,..., R_j) \prod_{i=1}^{j} \zeta_{m_{R_i}}(R_i) \tag{2.43}$$

Formula (2.43) is a straightforward consequence of the Möbius inversion formula

$$\Psi_\Gamma(m_\Gamma) = (-1)^{|\Gamma|} \sum_{\gamma \subset \Gamma} (-1)^{|\gamma|} H_\gamma^{b,d}(m_\gamma)$$

relating potentials to Hamiltonians.

We can now estimate from above the usual norm of the potential $\Psi(m) \equiv \{\Psi_\Gamma(m_\Gamma)\}_{\Gamma \subset \mathscr{A}}$, with $|\Gamma| > 1$

$$\|\Psi\| = \sum_{\Gamma:\, A_o \subset \Gamma} \sup_{m_\Gamma} |\Psi_\Gamma(m_\Gamma)|$$

as follows:

$$\|\Psi\| \leqslant \sum_{k \geqslant 1} \sum_{\substack{R_1,\ldots,R_k \in \mathscr{R}: \\ A_o \subset \bigcup \bar{R}_i}} |\varrho(R_1,\ldots,R_k)| \prod_{j=1}^{k} \sup_{m_j} |\zeta_{m_j}(R_j)| \tag{2.44}$$

From Propositions 2.1 and 2.2 and in particular from (2.38) and (2.39) we get that, for h and β sufficiently large, the r.h.s. of (2.44) is finite. The proof of the theorem is completed.

Remark. In the above argument we always assumed the external field h to be large only for simplicity. We could have covered the case of small field h and large (depending on h) inverse temperature β by simply replacing the 2×2 blocks in our construction by $l \times l$ blocks, with $l = C/h$ and $C > 4$, e.g., $C = 5$. With this choice one has in fact that the Hamiltonian of a single block B_i has a unique ground state identically equal to plus one, independently of the boundary conditions $\alpha_i^1 \cdots \alpha_i^4$. In particular, it follows that

$$\lim_{\beta \to \infty} \psi(\alpha_i^1 \cdots \alpha_i^4) = 0$$

and the convergence of the cluster expansion can again be proved.

Of course, in the above argument the fact that the unique ground state is the special configuration identically equal to plus one is completely irrelevant. Thus the above method is able to treat systems at low enough temperature having the property that the ground state in a large enough volume is unique and independent of the boundary conditions.

One may also want to consider much more general cases in which the thermodynamic parameters guarantee only a strong form of weak dependence of the finite-volume Gibbs measure on the boundary conditions, which we call *strong mixing* (see, e.g., ref. 11 and references therein).

This is the case, for example, of 2D ferromagnetic systems just above the critical point.[11] In such more general situations, even if the side of the blocks is large, the decimation over the odd blocks may not be enough to depress some long-range dependence in the doubly renormalized measure and one is forced to decimate further. One may stop the extra decimation until each surviving block, namely the blocks of the final block spin transformation, are separated from one another by at least one block of the decimation. We shall omit the details of these computations, which are quite similar to the ones exposed above.

3. AN EXAMPLE OF PERSISTENCE OF NON-GIBBSIANNESS UNDER DECIMATION

In this last section we briefly discuss another example of a measure μ on $\{0, 1\}^{\mathbf{Z}^2}$ with nicely decaying correlations functions, which is *non-Gibbsian* and remains such even after decimation on blocks of arbitrary side. The example comes from a model of random discrete-time dynamics introduced in ref. 12 and further analyzed in ref. 9.

The setting is as follows: to each point x in the lattice $\mathbf{Z}^2$ we associate an occupation variable $\sigma(x)$ with value 0 or 1; given a configuration $\sigma \in \{0, 1\}^{\mathbf{Z}^2}$, we then define its clusters as the maximal connected sets of sites in which the configuration σ is equal to one, where a set $C \subset \mathbf{Z}^2$ is connected if for any pair of sites $x, y \in C$ there exists a path $x_1, x_2, \ldots, x_n$ of sites in C such that

$$x_1 = x, \qquad x_n = y, \qquad \text{and} \qquad |x_i - x_{i+1}| = 1, \qquad i = 1, \ldots, n-1$$

With this position the dynamics goes as follows: given the configuration $\sigma_t \in \{0, 1\}^{\mathbf{Z}^2}$ at time t, in order to define the new configuration σ_{t+1} at time $t+1$, we first remove each cluster of σ_t independently of one another with probability 1/2; as a second step we create particles in each empty site independently with probability p.

For brevity we will refer to the first part of the updating as the annihilation of particles and to the second part as the creation of particles. Note that both processes occur simultaneously (i.e., the updating is parallel) and that the nontrivial interaction of the model is all contained in the killing process.

The above dynamics is similar to a model considered by Grannan and Swindle,[7] although in their model clusters disappear with a rate proportional to their size. The two main results that we need from refs. 12 and 9 are the following:

Theorem 3.1. (i) For p sufficiently mall there exists a unique invariant measure μ on $\{0, 1\}^{\mathbf{Z}^2}$ for the above dynamics and its truncated correlations decay faster than any inverse power (see Corollary 5.1 in ref. 9).

(ii) If Ω_N denotes the event that the cube Λ_N of side N centered at the origin is filled with particles, then there exists a constant c such that

$$\mu(\Omega_N) \geqslant \exp(-cN)$$

(see Theorem 5.1 in ref. 12).

In particular, part (ii) of the above theorem implies that the measure μ cannot be the Gibbs measure for any absolutely summable interaction, since it has *wrong* large deviations: an exponential of the surface instead of an exponential of the volume (see, e.g., ref. 4 for more details).

Remark. It is important to observe that the measure μ is non-Gibbsian for reasons which are very different from the ones behind the *non-Gibbsianness* of the block spin (without decimation) Ising model discussed in Section 2. There Gibbsianness is lost due to the fact that, conditioned to the event of having zero magnetization in each 2×2 block, the original spin model undergoes a phase transition.[4] Here, on the contrary, Gibbsianness is violated because the measure μ has zero relative entropy density with respect to the δ-measure concentrated on the configuration identically equal to plus one.[4] Clearly the latter is non-Gibbsian because it violates the nonnullity condition or absence of hard-core interaction (see ref. 4, 4.5.5 and 2.3.3). We actually suspect that μ itself violates the nonullity condition, but we do not have a formal proof of this.

Keeping in mind such a difference between the block spin Ising model and the present one, it is not entirely surprising that a decimation can suppress the phase transition in the first one and thus restore the Gibbsianness, while it is useless in this new situation, where the non-Gibbsianness is due to a much stronger and more rigid phenomenon.

We want now to apply the usual decimation described in Section 2 to the measure μ and show that it leads to a new measure μ^{d, b_l} which is again non-Gibbsian irrespective of the side l of the blocks on which the decimation acts.

As before, let us consider the partition of $\mathbf{Z}^2$ into $l \times l$ square blocks Q_i of integer side l and let us decompose the lattice $\mathbf{Z}^2$ into the two odd and even sublattices $\mathbf{Z}^2_e$ and $\mathbf{Z}^2_o$, namely

$$\mathbf{Z}^2 = \mathscr{A} \cup \mathscr{B} \tag{3.1}$$

where $\mathscr{A} = \bigcup_i A_i$ is the set of the even blocks and $\mathscr{B} = \bigcup_i B_i$ is the set of the odd blocks (see Section 2 for more details). Let us also call μ^{d,b_l} the projection or relativization of the measure μ to $\mathscr{A}$. Then we have:

Theorem 3.2. For any value of the decimation parameter l the measure μ^{d,b_l} is not Gibbsian for any absolutely summable interaction.

Proof. Let us denote by Ω_N^A the event that all the even blocks A contained inside the cube Λ_N of side N centered at the origin are filled with particles. Then we trivially have

$$\mu^{d,b_l}(\Omega_N^A) = \mu(\Omega_N^A) \geqslant \mu(\Omega_N) \geqslant \exp(-cN)$$

Thus also the decimated measure μ^{d,b_l} has wrong large deviations and the result follows.

Remark. We want to remark that, by the same argument used before for the decimation in one of the two L-block sublattices, it is immediate to prove that the non-Gibbsianness of our measure μ persists under any *extensive* decimation; namely a decimation such that the surviving spins, even though they can be very sparse, have a well-defined volume density.

Remark. We notice that the above example is a particular case of a general situation which can be described as follows: We start from a measure μ having zero relative entropy density w.r.t. a delta measure; so by general arguments,[4] μ is not Gibbsian.

We then apply to μ any deterministic renormalization-group transformation T (decimation, averaging, majority rule over odd blocks,...; see ref. 4). Lemma 3.3 in ref. 4 implies that the transformed measure $T\mu$ still has zero relative entropy density w.r.t. a new delta measure; thus $T\mu$ is non-Gibbsian as well.

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